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GENERALIZED NORMAL FORMS FOR POLYNOMIAL VECTOR FIELDS ☆

Jesús PALACIÁN, Patricia YANGUAS

Departamento de Matemática e Informática, Universidad Pública de Navarra, 31006 Pamplona, Navarra, Spain

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ABSTRACT. – A method to obtain formal symmetries of polynomial vector fields with non-null linear part is presented. We show that, under certain conditions, the symmetries of the linear components can be extended to higher degree terms by means of adequate changes of variables. The approach is based on a generalization of the Normal Form Theorem for vector fields. Lie transformations for ordinary differential equations are used to extend the symmetries to any order. Reduced phase spaces are constructed by making use of the first integrals associated to the linear part of the new symmetry. For Hamiltonian vector fields, as the formal symmetries become integrals of motion, this procedure produces a reduction of the number of degrees of freedom at least by one. We illustrate the technique with some examples for Hamiltonian and non-Hamiltonian vector fields. © 2001 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

In this paper we deal with dynamical systems of the type:

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = A\boldsymbol{x}(t) + \sum_{i=1}^{L} \frac{\varepsilon^i}{i!} \boldsymbol{F}_i\big(\boldsymbol{x}(t)\big), \tag{1.1}$$

where t represents the time variable, $\boldsymbol{x} \in \boldsymbol{R}^m$, A is an $(m \times m)$-matrix with constant coefficients with physical dimensions [time^{-1}], ε stands for a dimensionless small parameter, $\boldsymbol{F}_i$ is a vector field with m components and each component is a homogeneous polynomial of degree $i+1$. Note that L can be interpreted as the degree reached by the Taylor development of an analytic vector field, thus it can be infinity.

Many dynamical systems are modelled by a system of ordinary differential equations of the type (1.1) which is formed by the sum of a linear part plus a small polynomial perturbation. Moreover, the perturbation is a sum of non-linear vector fields which begins with quadratic polynomials.

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E-mail addresses: palacian@unavarra.es (J. Palacián), yanguas@unavarra.es (P. Yanguas).

These equations appear in the context of Dynamical Systems Theory; for instance in stability and bifurcation analysis of equilibrium points, periodic orbits or singularity theories. Besides, applications are encountered in several fields such as Biology (predator-prey models, see [27]), Medicine (see [15] and references therein), Engineering (van der Pol-like and Duffing-like equations, see [11]) or Astrodynamics (potential functions modelling elliptical galaxies, cf. [7, 36]).

Analytical methods to deal with dynamical systems like (1.1) are based on the fact that these differential equations can be understood as perturbations of the principal part, $A\boldsymbol{x}(t)$. In this context of Perturbation Theory our purpose is to reduce the problem to a simpler one by means of adequate changes of variables.

The first step in this direction was given by Poincaré in [26]. He considered the problem of reducing (1.1) to a system of linear equations

$$\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = A\, \boldsymbol{y}(t),$$

that is, linearizing (1.1), by defining the formal change of variables $\boldsymbol{y}(t) = \boldsymbol{\varrho}(\boldsymbol{x}(t); \varepsilon) = \boldsymbol{x}(t) + \cdots$. Concretely, he found a solution in the case where A is diagonalizable and its eigenvalues $\lambda_1, \ldots, \lambda_m$ are non-resonant, i.e.: $\lambda_j \neq \sum k_i\, \lambda_i$ for $j = 1, \ldots, m$ and for all integer vectors $\boldsymbol{k} = (k_1, \ldots, k_m)$ with $k_i \geqslant 0$ and $|k| = k_1 + \cdots + k_m \geqslant 2$. He also proved that, if in addition to the above, the eigenvalues of A, considered in the complex plane, lie strictly to one side of a line through the origin of the complex plane, then the formal series which gives the change of variables converges (for a proof see Theorem 13.2, pp. 179–180 of reference [31]).

The main contribution after Poincaré's is the Normal Form Theorem given by Meyer [18]. He reduces system (1.1) to a simpler one by an adequate change of variables, for systems where A_S (the semisimple part of A) is not zero. In this case, the vector field $A_S\, \boldsymbol{y}$ becomes a formal symmetry of the transformed system. The results obtained by Meyer were extended by Elphick et al. [9].

The purpose of this paper is to extend the results of this theorem. We construct a formal change of variables such that the transformed differential equation defines a system with a new symmetry (up to a certain order of approximation) for matrices A whose semisimple part A_S can be zero. With this method it is also possible to construct several symmetries for the same system. As in the case of the Normal Form Theorem, the implementation of this method makes use of Lie transformations. An adaptation of our procedure for Hamiltonian systems can be found in [23]. In reference [25] we propose the method for the case of systems of differential equations without giving the proof of the theorems but illustrating the results with some examples.

It is already known [4,32] that the calculation of a normal form accomplishes an effective reduction of a departure system provided that $A_S \neq 0$. Let $0 \leqslant s \leqslant m$ be the number of functionally-independent polynomial first integrals associated to the linear system $\mathrm{d}\boldsymbol{y}(t)/\mathrm{d}t = A_S\, \boldsymbol{y}(t)$. Note first that the number of linearly-independent polynomial first integrals of the linear system is $r \geqslant s$ and that r can be bigger than, equal to or smaller than m. Indeed, the r first integrals are used to build the "main part" of the normal form. Specifically, it has been proven (the so-called Splitting Lemma, see [10] and references therein) that the normalized vector field can be split into two subsystems: one of the subsystems is of dimension $m - s$ and is defined on an A_S-invariant space. The other subsystem has dimension r; it is also defined on an A_S-invariant space and contains the fundamental dynamics of the departure system. Moreover, in this second space all the equation is expressed totally in terms of the r first integrals although the corresponding phase space has dimension s. Actually this can be done due to the fact that the vector field $A_S\boldsymbol{y}$ is a linear symmetry of the normal form system. Note that the systems of equations coming out of the splitting are smooth provided that the normal form is smooth. The Splitting Lemma can be applied as well to non-polynomial first integrals (whether $s = 0$) but then the smoothness of the reduction process cannot be guaranteed.

We shall use the Splitting Lemma in our setting, showing how the r first integrals associated to the linear part of the new symmetry must be employed to construct the main part of the normalized system (of dimension r). Besides, we shall see that the reduced phase space has to be determined by the s functionally-independent first integrals. The r linearly-independent first integrals are indeed the generators of the normal form and the coordinates of the reduced phase space. We shall focus on the Hamiltonian case as in this special situation we will only have an r-dimensional system of equations defined on a phase space of dimension s.

We do not deal with the convergence of the transformations in this paper, though it is well known that transformations based on normal form techniques diverge, in general. In this respect, several approaches have been given since the pioneering work by Bruno [4]. Concretely, for two-dimensional polynomial vector fields of type (1.1), Bruno and Walcher [5] give necessary and sufficient conditions to assure convergent transformations. Basically, a convergent transformation is guaranteed if there is a nontrivial local one-parameter group of symmetries. See also reference [34].

The idea of approximating symmetries for vector fields is not new. Cicogna and Gaeta, cf. [6], generalize the procedures based on normal forms to extend the linear symmetries of a system $\mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t = A\boldsymbol{x}(t)$ to the non-linear part, up to a certain order i for which the corresponding part of the generating function is not longer a vector field with polynomial components. In this direction, we propose to go beyond by allowing the generating function to have rational components. As the denominators of these rational functions can vanish we have to exclude some values from the domain of validity of such transformations, but this is the only way of introducing (formal) asymptotic symmetries for systems of differential equations whose linear parts have a nilpotent matrix, e.g., $A_S = 0$. We shall illustrate these features with some examples.

In normal form computations, much attention is commonly put in the qualitative behaviour on the centre manifold of the equation, and only the normal form on this invariant manifold is of interest. The reason is that the coefficients of the transformations can be used to determine the stability of critical points and for bifurcation problems. Other qualitative aspects concern the calculation of equilibrium points and periodic orbits from the (generalized) normal forms and their connection to the periodic orbits and n-dimensional tori of the original system ($n \leqslant m$). We are not going to treat this issue here. We refer the reader to references [9,33] for the computation focused on centre manifolds and to reference [12] for a qualitative analysis of periodic orbits.

The paper is divided into five sections. Section 2 recalls the Normal Form Theorem and contains the generalization of normal forms. We analyze the hypotheses required to obtain formal symmetries with generating functions formed by polynomials or by rational functions. In Section 3 we translate the results to the Hamiltonian context, pointing out the peculiarities of the symplectic case. In Section 4 we describe the geometrical and computational aspects of the reduction after the application of the generalized normal form, dealing with the invariants and reduced phase spaces. Finally in Section 5 we illustrate the technique with two examples of differential systems.

2. Constructing formal symmetries

In this section we first give theoretical results related to the construction of formal symmetries for analytic vector fields not necessarily of polynomial type. Then, as a particular case, we will concentrate on polynomial vector fields.

2.1. Lie transformations for vector fields

Meyer's approach to the calculation of formal symmetries is based on Lie transformations. The work of Meyer in this direction [19] is based on previous works by Deprit [8], Kamel [14] and Henrard [13]. In [8], Deprit applies Lie transformations to Hamiltonian systems, whereas references [14] and [13] deal with the generalization of this technique to differential equations. Finally [19] presents a Lie transformations treatment in the context of tensor fields. We start by recalling the Lie transformations method applied to analytic vector fields.

Let us consider the system:

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{F}_0\big(\boldsymbol{x}(t)\big) + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \boldsymbol{F}_i\big(\boldsymbol{x}(t)\big), \tag{2.1}$$

where t represents the time variable, $\boldsymbol{x} \in \boldsymbol{R}^m$, ε stands for a dimensionless small parameter and $\boldsymbol{F}_i$, $i \geqslant 0$, is a vector field with m components, which are analytic functions in $\boldsymbol{x}$. We define by $[\cdot\,,\,\cdot]$ the Lie bracket of two vector fields $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ in $\boldsymbol{R}^m$, that is, $[\boldsymbol{g}_1, \boldsymbol{g}_2] = (\partial \boldsymbol{g}_1/\partial \boldsymbol{x})\, \boldsymbol{g}_2 - (\partial \boldsymbol{g}_2/\partial \boldsymbol{x})\, \boldsymbol{g}_1$.

Let us describe the typical algorithm of Lie transformations. An analytic vector field (2.1) depending on a small parameter ε, is transformed into another vector field

$$\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = \boldsymbol{G}_0\big(\boldsymbol{y}(t)\big) + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \boldsymbol{G}_i\big(\boldsymbol{y}(t)\big), \tag{2.2}$$

where $\boldsymbol{G}_0(\boldsymbol{y}(t)) \equiv \boldsymbol{F}_0(\boldsymbol{x}(t))$, through a generating function

$$\boldsymbol{W}(\boldsymbol{x};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} \boldsymbol{W}_{i+1}(\boldsymbol{x}),$$

following the recursive formula

$$\boldsymbol{F}_i^{(j)} = \boldsymbol{F}_{i+1}^{(j-1)} + \sum_{k=0}^{i} \binom{i}{k} \big[\boldsymbol{F}_{i-k}^{(j-1)}, \boldsymbol{W}_{k+1}\big], \tag{2.3}$$

with $i \geqslant 0$, $j \geqslant 1$. Besides, $\boldsymbol{F}_i^{(0)} \equiv \boldsymbol{F}_i$ and $\boldsymbol{F}_0^{(i)} \equiv \boldsymbol{G}_i$ for all $i \geqslant 0$.

Note that $\boldsymbol{W}(\boldsymbol{x};\varepsilon)$ is conserved under the transformation and thus, it can also be expressed as $\boldsymbol{W}(\boldsymbol{y};\varepsilon)$, that is, $\boldsymbol{W}(\boldsymbol{x};\varepsilon) \equiv \boldsymbol{W}(\boldsymbol{y};\varepsilon)$.

Hence, equation (2.3) yields the partial differential identity:

$$\mathcal{L}_{\boldsymbol{F}_0}(\boldsymbol{W}_i) + \boldsymbol{G}_i = \widetilde{\boldsymbol{F}}_i, \tag{2.4}$$

where $\widetilde{\boldsymbol{F}}_i$ collects all the terms known from the previous order. In this identity, called the homology equation, $\boldsymbol{W}_i$ and $\boldsymbol{G}_i$ must be determined according to the specific requirements of the Lie transform one performs. Besides, $\mathcal{L}_{\boldsymbol{F}_0}$ denotes the Lie operator associated to the Lie bracket of two vector functions, i.e., given two vector fields $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$: $\mathcal{L}_{\boldsymbol{g}_2}(\boldsymbol{g}_1) = [\boldsymbol{g}_1,\ \boldsymbol{g}_2]$.

The transformation $\boldsymbol{X}(\boldsymbol{y};\varepsilon)$ relates the "old" variables $\boldsymbol{x}$ with the "new" ones $\boldsymbol{y}$ and is a near-identity change of variables. Explicitly, the direct change is given by:

$$\boldsymbol{x} = \boldsymbol{y} + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \mathcal{L}^i_{\boldsymbol{W}}(\boldsymbol{y}), \tag{2.5}$$

where the Lie operator applied to a vector $\boldsymbol{y}$ means that it is applied to each component of $\boldsymbol{y}$. Besides, the notation $\mathcal{L}^i_{\boldsymbol{W}}$ refers to the application of $\mathcal{L}_{\boldsymbol{W}}$ i times, that is, $\mathcal{L}^i_{\boldsymbol{W}}(\boldsymbol{y}) = \mathcal{L}_{\boldsymbol{W}}(\mathcal{L}^{i-1}_{\boldsymbol{W}}(\boldsymbol{y}))$, if $i \geqslant 2$. Consequently, equation (2.5) gives the set of variables $\boldsymbol{x}$ in terms of $\boldsymbol{y}$ with the use of the generating function $\boldsymbol{W}$. Similar formulae can be used to obtain the inverse transformation, which is:

$$\boldsymbol{y} = \boldsymbol{x} + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \mathcal{L}^i_{-\boldsymbol{W}}(\boldsymbol{x}), \tag{2.6}$$

where $\mathcal{L}_{-\boldsymbol{W}}$ refers to the Lie operator $\mathcal{L}_{-\boldsymbol{W}} : \boldsymbol{g} \to [\boldsymbol{W}, \boldsymbol{g}]$. Besides, the notation $\mathcal{L}^i_{-\boldsymbol{W}}$ refers to the application of the inverse Lie operator $\mathcal{L}_{-\boldsymbol{W}}$ i times.

Note that equation (2.5) can be used to transform any function expressed in the old variables $\boldsymbol{x}$ as a function of the new variables $\boldsymbol{y}$. Similarly, equation (2.6) is used to transform any function in $\boldsymbol{y}$ as a function of $\boldsymbol{x}$.

The above method is formal in the sense that the convergence of the various series is not discussed. Moreover, the series diverge in many applications. However, the first orders of the transformed system can give interesting information and the process can be stopped at a certain order M. This means that these terms of the series are useful to construct both the transformed vector field and the generating function since they are unaffected by the divergent character of the whole process. In these circumstances, the General Perturbation Theorem applies.

THEOREM 2.1 (General Perturbation Theorem (Meyer)). – *Let $M \geqslant 1$ be given, let $\{\mathcal{P}_i\}_{i=0}^M$, $\{\mathcal{Q}_i\}_{i=1}^M$ and $\{\mathcal{R}_i\}_{i=1}^M$ be sequences of vector spaces of analytic functions in $\boldsymbol{x} \in \boldsymbol{R}^m$ defined on a common domain Ω in $\boldsymbol{R}^m$ with the following properties*:

(i) $\mathcal{Q}_i \subseteq \mathcal{P}_i,\ i = 1, \ldots, M$;
(ii) $\boldsymbol{F}_i \in \mathcal{P}_i,\ i = 0, 1, \ldots, M$;
(iii) $[\mathcal{P}_i, \mathcal{R}_j] \subseteq \mathcal{P}_{i+j},\ i + j = 1, \ldots, M$;
(iv) *for any $\boldsymbol{D} \in \mathcal{P}_i$, $i = 1, \ldots, M$, one can find $\boldsymbol{E} \in \mathcal{Q}_i$ and $\boldsymbol{H} \in \mathcal{R}_i$ such that*:

$$\boldsymbol{E} = \boldsymbol{D} + [\boldsymbol{F}_0, \boldsymbol{H}].$$

Then, there is an analytic vector field $\boldsymbol{W}$,

$$\boldsymbol{W}(\boldsymbol{x};\varepsilon) = \sum_{i=0}^{M-1} \frac{\varepsilon^i}{i!}\, \boldsymbol{W}_{i+1}(\boldsymbol{x}),$$

with $\boldsymbol{W}_i \in \mathcal{R}_i$, $i = 1, \ldots, M$, such that the change of variables $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{y};\varepsilon)$ is the general solution of:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\varepsilon} = \frac{\partial \boldsymbol{W}}{\partial \boldsymbol{x}}(\boldsymbol{x};\varepsilon),$$
$$\boldsymbol{x}(0) = \boldsymbol{y},$$

and transforms the convergent vector field

$$\boldsymbol{F}(\boldsymbol{x};\varepsilon)=\sum_{i=0}^{\infty}\frac{\varepsilon^i}{i!}\boldsymbol{F}_i(\boldsymbol{x}),$$

to the convergent vector field

$$\boldsymbol{G}(\boldsymbol{y};\varepsilon)=\sum_{i=0}^{M}\frac{\varepsilon^i}{i!}\boldsymbol{G}_i(\boldsymbol{y})+\mathcal{O}\bigl(\varepsilon^{M+1}\bigr),$$

with $\boldsymbol{G}_i\in\mathcal{Q}_i$, $i=1,\dots,M$.

Proof. – See reference [19]. □

Remark 1. – The latter theorem can be extended for time-dependent ordinary differential equations $\mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t=\boldsymbol{F}(t,\boldsymbol{x};\varepsilon)$ by making use of Lie transformations procedures suitable for non-autonomous equations. For an effective application of the algorithm one has to introduce a remainder function as a formal series in the small parameter. Besides, the sequences of vector spaces $\{\mathcal{P}_i\}_{i=0}^M$, $\{\mathcal{Q}_i\}_{i=1}^M$ and $\{\mathcal{R}_i\}_{i=1}^M$ must be defined on a common domain Ω in $\boldsymbol{R}\times\boldsymbol{R}^m$ and one has to add the sequence $\{\dot{\mathcal{R}}_i\}_{i=1}^M$ (formed by vector spaces of all derivatives of functions in $\mathcal{R}_i$). Besides, $[\mathcal{P}_i,\dot{\mathcal{R}}_j]\subseteq\mathcal{P}_{i+j}$ for $i+j=1,\dots,M$ and for any $\boldsymbol{D}\in\mathcal{P}_i$, $i=1,\dots,M$, one can find $\boldsymbol{E}\in\mathcal{Q}_i$ and $\boldsymbol{H}\in\mathcal{R}_i$ such that $\boldsymbol{E}=\boldsymbol{D}+[\boldsymbol{F}_0,\boldsymbol{H}]-\partial\boldsymbol{H}/\partial t$. Thus, there is a vector field $\boldsymbol{W}$ which generates a near-identity change of variables $\boldsymbol{x}\to\boldsymbol{y}$. Besides, the remainder of the transformation is determined step by step through the generating function (see [19] for details). In that context, a particular situation is the method of averaging by Bogoliubov and Mitropolskii (see cf. [27] and references therein). For accomplishing the averaging procedure one has to start the Lie transformation with $\boldsymbol{F}_0(t,\boldsymbol{x})=\boldsymbol{0}$ and each (time-independent) $\boldsymbol{G}_i$ is calculated as an average over the period $T>0$, specifically, $\boldsymbol{G}_i(\boldsymbol{y})=\frac{1}{T}\int_0^T\widetilde{\boldsymbol{F}}_i(\tau,\boldsymbol{y})\,\mathrm{d}\tau$. Besides, the part of the generating function corresponding to the order i should be calculated as the quadrature $\boldsymbol{W}_i(t,\boldsymbol{y})=\int[\widetilde{\boldsymbol{F}}_i(t,\boldsymbol{y})-\boldsymbol{G}_i(\boldsymbol{y})]\,\mathrm{d}t$.

Remark 2. – Estimates of the error committed by the application of the General Perturbation Theorem can be obtained from the theory developed for the method of averaging. In that sense, if we call $\boldsymbol{F}^*(\boldsymbol{y};\varepsilon)=\boldsymbol{F}(\boldsymbol{x}(\boldsymbol{y};\varepsilon);\varepsilon)$, then one can conclude that by using an adequate norm, $\|\boldsymbol{F}^*(\boldsymbol{y};\varepsilon)-\boldsymbol{G}(\boldsymbol{y};\varepsilon)\|=\mathcal{O}(\varepsilon^{M+1})$ on a time-scale $1/\varepsilon$, see reference [27] for many examples dealing with estimates.

Remark 3. – Approximate first integrals can be obtained by computing asymptotic vector integrating factors and it can be put in a General Perturbation Theory frame (see the paper by van Horssen [30]). However we do not enter this subject in the general context although we shall go back to this for polynomial vector fields in Section 4 when dealing with the reduced equation.

Now we are ready to extend Theorem 2.1 for the construction of formal symmetries for vector fields. For this we give a corollary in which we add an extra hypothesis.

COROLLARY 2.2. – *Let* $M\geqslant 1$ *be given, let* $\{\mathcal{P}_i\}_{i=0}^M$, $\{\mathcal{Q}_i\}_{i=1}^M$ *and* $\{\mathcal{R}_i\}_{i=1}^M$ *be sequences of vector spaces of analytic functions in* $\boldsymbol{x}\in\boldsymbol{R}^m$ *defined on a common domain* Ω *in* $\boldsymbol{R}^m$ *and let* $\boldsymbol{T}\equiv\boldsymbol{T}(\boldsymbol{x})$ *be a vector field in some* $\{\mathcal{P}_i\}_{i=0}^M$ *with the following properties*:

(i) $\mathcal{Q}_i\subseteq\mathcal{P}_i$, $i=1,\dots,M$;
(ii) $\boldsymbol{F}_i\in\mathcal{P}_i$, $i=0,1,\dots,M$;
(iii) $[\mathcal{P}_i,\mathcal{R}_j]\subseteq\mathcal{P}_{i+j}$, $i+j=1,\dots,M$;

(iv) *for any* $\boldsymbol{D} \in \mathcal{P}_i$, $i = 1, \ldots, M$, *one can find* $\boldsymbol{E} \in \mathcal{Q}_i$ *and* $\boldsymbol{H} \in \mathcal{R}_i$ *such that*

$$\boldsymbol{E} = \boldsymbol{D} + [\boldsymbol{F}_0, \boldsymbol{H}] \quad \textit{and} \quad [\boldsymbol{T}, \boldsymbol{E}] = \boldsymbol{0}.$$

Then, there is an analytic vector field $\boldsymbol{W}$:

$$\boldsymbol{W}(\boldsymbol{x}; \varepsilon) = \sum_{i=0}^{M-1} \frac{\varepsilon^i}{i!} \boldsymbol{W}_{i+1}(\boldsymbol{x}),$$

with $\boldsymbol{W}_i \in \mathcal{R}_i$, $i = 1, \ldots, M$, *such that the change of variables* $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{y}; \varepsilon)$ *is the general solution of*:

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\varepsilon} &= \frac{\partial \boldsymbol{W}}{\partial \boldsymbol{x}}(\boldsymbol{x}; \varepsilon), \\ \boldsymbol{x}(0) &= \boldsymbol{y}, \end{aligned}$$

and transforms the convergent vector field

$$\boldsymbol{F}(\boldsymbol{x}; \varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} \boldsymbol{F}_i(\boldsymbol{x}),$$

to the convergent vector field

$$\boldsymbol{G}(\boldsymbol{y}; \varepsilon) = \sum_{i=0}^{M} \frac{\varepsilon^i}{i!} \boldsymbol{G}_i(\boldsymbol{y}) + \mathcal{O}\big(\varepsilon^{M+1}\big),$$

with $\boldsymbol{G}_i \in \mathcal{Q}_i$ *and* $[\boldsymbol{G}_i, \boldsymbol{T}] = \boldsymbol{0}$, $i = 1, \ldots, M$. *Besides, if* $[\boldsymbol{F}_0, \boldsymbol{T}] = \boldsymbol{0}$ *then* $\boldsymbol{T} \equiv \boldsymbol{T}(\boldsymbol{y})$ *is a formal symmetry of* $\boldsymbol{G}$.

Proof. – Note that the difference between this result and Theorem 2.1 is that here we introduce the vector field $\boldsymbol{T}$. Then, condition (iv) of Theorem 2.1 is slightly modified in the sense that we also require that functions $\boldsymbol{E} \in \mathcal{Q}_i$ satisfy $[\boldsymbol{T}, \boldsymbol{E}] = \boldsymbol{0}$. According to Theorem 2.1, $\boldsymbol{G}_i \in \mathcal{Q}_i$, then the additional thesis $[\boldsymbol{G}_i, \boldsymbol{T}] = \boldsymbol{0}$ is satisfied. □

2.2. The Normal Form Theorem for the general equilibrium

In the polynomial context, Meyer [18] generalized Poincaré's results by giving the Normal Form Theorem, which is a consequence of the General Perturbation Theorem (Theorem 2.1). The work by Meyer was then extended by Elphick et al. [9]. See also this paper for a proof of Meyer's main theorem as well as for many examples of normal forms computations. The idea is to find the normal form of (1.1), that is, to reduce this system to an "equivalent" one:

$$\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = A\,\boldsymbol{y}(t) + \sum_{i=1}^{M} \frac{\varepsilon^i}{i!} \boldsymbol{G}_i\big(\boldsymbol{y}(t)\big) + \mathcal{O}\big(\varepsilon^{M+1}\big), \tag{2.7}$$

where $\boldsymbol{y} \in \boldsymbol{R}^m$, $M \geqslant 1$ and $\boldsymbol{G}_i$ are "simpler" than $\boldsymbol{F}_i$ for $i = 1, \ldots, M$.

We denote by $A = A_S + A_N$ the Jordan decomposition of the non-null real square matrix A into its semisimple and nilpotent parts, respectively. Matrix A^t represents the transpose matrix of A (which is, indeed, its adjoint matrix for the standard inner product in $\boldsymbol{R}^m$). The notation $\mathcal{L}_A$ represents the linear operator associated to A, such that, for any vector field $\boldsymbol{g}$ defined in a domain of $\boldsymbol{R}^m$, $\mathcal{L}_A(\boldsymbol{g}) = [\boldsymbol{g}, A\boldsymbol{x}]$.

Before recalling the Normal Form Theorem we need the following lemmas, which appear in [9] (Section 2.2, pp. 101–103). See also reference [20].

LEMMA 2.3 (Fredholm alternative (Elphick et al.)). – *Let $\boldsymbol{V}$ be a finite-dimensional inner product space with inner product $(\cdot\,,\,\cdot)$. Consider $F:\boldsymbol{V} \to \boldsymbol{V}$ as a linear transformation, and F^* its adjoint, so $(F\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, F^*\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$. Then, $\boldsymbol{V} = \ker(F^*) \oplus \operatorname{im}(F)$.*

This indicates that one can split the vectors of $\boldsymbol{V}$ as sum of two components, one belonging to the kernel of F^* and the other to the image of F.

LEMMA 2.4 (Elphick et al.). – *Let us identify $\boldsymbol{V}$ with $\boldsymbol{R}^m$ and choose the standard inner product in $\boldsymbol{R}^m$, then the adjoint matrix of a real $(m \times m)$-matrix A is $A^* = A^t$. Let $\mathcal{P}_i$ be the vector space whose elements are m-dimensional vector fields where each component is a homogeneous polynomial of degree $i+1$ in $\boldsymbol{x} \in \boldsymbol{R}^m$. Let $\mathcal{Z}$ denote the set of all m-tuples of non-negative integers. Define $|k| = k_1 + \cdots + k_m$ and $\boldsymbol{x}^k = x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$, where $k = (k_1, \ldots, k_m) \in \mathcal{Z}$, $\boldsymbol{x} \in \boldsymbol{R}^m$. Let $\boldsymbol{S}$ and $\boldsymbol{T}$ be two vector fields of $\mathcal{P}_i$ such that their components are $S_j = \sum_{|k|=i+1} s_{j\,k}\, \boldsymbol{x}^k$ and $T_j = \sum_{|h|=i+1} t_{j\,h}\, \boldsymbol{x}^h$, $1 \leqslant j \leqslant m$, that is: homogeneous polynomials of degree $i+1$. Define now an inner product $\langle \cdot\,,\,\cdot \rangle$ on $\mathcal{P}_i$ such that $\langle \boldsymbol{S}, \boldsymbol{T} \rangle$ is a vector field in $\mathcal{P}_i$ whose components are given by $\sum_{|k|=i+1} k!\, s_{j\,k}\, t_{j\,k}$, $1 \leqslant j \leqslant m$. Hence, the identity $\langle \boldsymbol{S}, \mathcal{L}_A(\boldsymbol{T}) \rangle = \langle \mathcal{L}_{A^t}(\boldsymbol{S}), \boldsymbol{T} \rangle$ holds. So the adjoint of $\mathcal{L}_A$ with respect to $\langle \cdot, \cdot \rangle$ is $\mathcal{L}_{A^t}$.*

Lemmas 2.3 and 2.4 are the key to decompose the vector space $\mathcal{P}_i$ into the direct sum of two subspaces, using as the scalar product, the inner product $\langle \cdot\,,\,\cdot \rangle$ and as kernel and image, the ones induced by the Lie operators $\mathcal{L}_A$ and $\mathcal{L}_{A^t}$. For the proofs of the above results the reader is referred to [9]. Now we can state the Normal Form Theorem for the general equilibrium.

THEOREM 2.5 (Normal Form Theorem (Meyer)). – *Given system* (1.1) *there is a formal transformation $\boldsymbol{\varrho}: \boldsymbol{R}^m \to \boldsymbol{R}^m$ such that $\boldsymbol{y}(t) = \boldsymbol{\varrho}(\boldsymbol{x}(t); \varepsilon) = \boldsymbol{x}(t) + \cdots$ which transforms it into* (2.7)*, where each term $\boldsymbol{G}_i$, $i = 1, \ldots, M$, belongs to the kernel of the operator $\mathcal{L}_{A^t}$, that is*:

$$\left[\boldsymbol{G}_i, A^t \boldsymbol{y}\right] = \boldsymbol{0}, \quad i = 1, \ldots, M. \tag{2.8}$$

Proof. – The reader is referred to [18,9]. □

The transformation is analytic since we truncate the change of variables at an order M. Besides, by virtue of Theorem 2.1 and the fact that $\ker \mathcal{L}_{A^t} \subset \ker \mathcal{L}_{A_S}$ (see reference [18]), one has that $[\boldsymbol{G}_i, A_S \boldsymbol{y}] = \boldsymbol{0}$, $i = 1, \ldots, M$. In addition to that, as A_S and A_N commute (i.e., $A_S A_N = A_N A_S$) then $[A\boldsymbol{y}, A_S\, \boldsymbol{y}] = \boldsymbol{0}$. Both facts imply that $A_S \boldsymbol{y}$ becomes a formal symmetry of the system given by (2.7).

However, if $A_S = 0$, the application of the Normal Form Theorem does not guarantee the introduction of a formal symmetry (although the transformed system is simpler than the original one). Then, we should follow a different strategy in order to simplify the system by introducing a formal symmetry. Moreover, even if $A_S \neq 0$, the Normal Form Theorem does not permit to simplify the initial system further or to obtain different symmetries, other than A_S.

Remark 4. – If the nilpotent part of A is not zero, the normal form system can be refined in the sense that a second reduction can be executed. In this case the second normal form would have the vector field $A_N \boldsymbol{y}$ as a new formal symmetry. This idea was introduced by van der Meer [28,29] for Hamiltonian systems and has been generalized thereafter for polynomial vector fields (see [2] and references therein). We can put this refinement procedure in the context of calculating formal symmetries we introduce in next subsection, as it is a manner of introducing a second symmetry, as the (first) normal form enjoys $A_S\boldsymbol{y}$ as a formal symmetry.

2.3. Generalized normal forms

In the following we will describe the way of transforming a system like (1.1) to a system of the type (2.7) so that this latter enjoys a linear symmetry $T\boldsymbol{y}$ for a given $(m \times m)$-matrix T. We shall achieve this goal by making use of Lie transformations.

Let T be an arbitrary $(m \times m)$-matrix. The construction of (2.7) is made step by step. In each order $i = 1, \ldots, M$ we have to calculate $\boldsymbol{G}_i$ and another vector field $\boldsymbol{W}_i$ which corresponds to the i-th term of the so-called generating function of the transformation. For this purpose one has to solve the homology equation:

$$\mathcal{L}_A(\boldsymbol{W}_i) + \boldsymbol{G}_i = \widetilde{\boldsymbol{F}}_i, \tag{2.9}$$

where $\widetilde{\boldsymbol{F}}_i$ denotes the vectors computed in the previous steps. We try to find $\boldsymbol{G}_i$ such that $[\boldsymbol{G}_i, T\boldsymbol{y}] = \boldsymbol{0}$. For that, we split $\widetilde{\boldsymbol{F}}_i = \widetilde{\boldsymbol{F}}_i^{\#} + \widetilde{\boldsymbol{F}}_i^{\&}$, where $\mathcal{L}_T(\widetilde{\boldsymbol{F}}_i^{\#}) = [\widetilde{\boldsymbol{F}}_i^{\#}, T\boldsymbol{y}] = \boldsymbol{0}$. Then, we identify $\boldsymbol{G}_i = \widetilde{\boldsymbol{F}}_i^{\#}$ and $\widetilde{\boldsymbol{F}}_i^{\&} = \widetilde{\boldsymbol{F}}_i - \widetilde{\boldsymbol{F}}_i^{\#}$. The vector $\boldsymbol{W}_i$ must be a solution of the system of partial differential equations $\mathcal{L}_A(\boldsymbol{W}_i) = \widetilde{\boldsymbol{F}}_i^{\&}$.

If, in addition to the above, T has been chosen so that it commutes with A, then $[A\boldsymbol{y}, T\boldsymbol{y}] = \boldsymbol{0}$ and $T\boldsymbol{y}$ is a symmetry of (2.7). Thus, one reduces the original system by the computation of a formal symmetry. Note that this approach enlarges the Normal Form Theorem since one does not need to take $A^t\boldsymbol{y}$ as the symmetry to be extended in order to reduce system (1.1) but any other vector field $T\boldsymbol{y}$. The following theorem formalizes what has been exposed in the previous paragraphs.

THEOREM 2.6. – *Let $M \geqslant 1$ be given. Let $\{\mathcal{P}_i\}_{i=0}^M$ be the sequence of the vector spaces of m-dimensional vector fields whose components are homogeneous polynomials of degree $i+1$ in $\boldsymbol{x} \in \boldsymbol{R}^m$. Let $\{\mathcal{Q}_i\}_{i=1}^M$ be the sequence of some subsets of the sets $\mathcal{P}_i$ and let $\{\mathcal{R}_i\}_{i=1}^M$ be the sequence of some vector spaces of m-dimensional vector fields whose components are rational functions, where the subtractions of the degrees of the numerators (homogeneous polynomials) by the degrees of the denominators (homogeneous polynomials) are $i+1$, for $i = 1, \ldots, M$. Let A and T be two non-null $(m \times m)$-matrices. Let $\Omega \in \boldsymbol{R}^m$ be the common domain where the sequences $\{\mathcal{P}_i\}_{i=0}^M$, $\{\mathcal{Q}_i\}_{i=1}^M$ and $\{\mathcal{R}_i\}_{i=1}^M$ are defined. Moreover, suppose that the following properties are satisfied:*

(i) $A\boldsymbol{x} \in \mathcal{P}_0$, $\boldsymbol{F}_i \in \mathcal{P}_i$, $i = 1, \ldots, M$;
(ii) $[\mathcal{P}_i, \mathcal{R}_j] \subseteq \mathcal{P}_{i+j}$, $i + j = 1, \ldots, M$;
(iii) *for any $\boldsymbol{D} \in \mathcal{P}_i$, $i = 1, \ldots, M$, there is $\boldsymbol{E} \in \mathcal{Q}_i$ and $\boldsymbol{H} \in \mathcal{R}_i$ such that:*

$$\boldsymbol{E} = \boldsymbol{D} + [A\boldsymbol{y}, \boldsymbol{H}] \quad \textit{and} \quad [T\boldsymbol{y}, \boldsymbol{E}] = \boldsymbol{0}.$$

Then, there is a vector field $\boldsymbol{W}$ whose m components are rational functions:

$$\boldsymbol{W}(\boldsymbol{x}; \varepsilon) = \sum_{i=0}^{M-1} \frac{\varepsilon^i}{i!} \boldsymbol{W}_{i+1}(\boldsymbol{x}),$$

with $\boldsymbol{W}_i \in \mathcal{R}_i$, $i = 1, \ldots, M$, such that the change of variables $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{y}; \varepsilon)$ is the general solution of the initial value problem:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\varepsilon} = \frac{\partial \boldsymbol{W}}{\partial \boldsymbol{x}}(\boldsymbol{x};\varepsilon),$$
$$\boldsymbol{x}(0) = \boldsymbol{y},$$

and transforms the convergent vector field:

$$\boldsymbol{F}(\boldsymbol{x};\varepsilon) = A\boldsymbol{x} + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!} \boldsymbol{F}_i(\boldsymbol{x})$$

into the vector field:

$$\boldsymbol{G}(\boldsymbol{y};\varepsilon) = A\boldsymbol{y} + \sum_{i=1}^{M} \frac{\varepsilon^i}{i!} \boldsymbol{G}_i(\boldsymbol{y}) + \mathcal{O}\big(\varepsilon^{M+1}\big),$$

with $\boldsymbol{G}_i \in \mathcal{Q}_i$, $i = 1, \ldots, M$, *such that each* $\boldsymbol{G}_i$ *is a vector field whose* m *components are homogeneous polynomials in* $\boldsymbol{y}$ *of degree* $i+1$ *with*

$$[\boldsymbol{G}_i, T\boldsymbol{y}] = \boldsymbol{0}$$

for $i = 1, \ldots, M$. *Moreover, if* $AT = TA$, *then* $T\boldsymbol{y}$ *is a formal symmetry of* $\boldsymbol{G}$.

Proof. – The result follows from Corollary 2.2. For $0 \leqslant i \leqslant M$ identify the sets $\mathcal{P}_i$ with the vector spaces of all vector fields whose components are homogeneous polynomials of degree $i+1$ in $\boldsymbol{x} \in \boldsymbol{R}^m$ and take $\mathcal{Q}_i$ as some subsets of $\mathcal{P}_i$, for $i = 1, \ldots, M$. Identify the sets $\mathcal{R}_i$ with the vector spaces of all vector fields whose components are rational functions where the subtractions of the degrees of the numerators (homogeneous polynomials) by the degrees of the denominators (homogeneous polynomials) are $i+1$. Then, by virtue of properties (i), (ii) and (iii), each $\boldsymbol{G}_i$ is a vector field in $\mathcal{Q}_i$ formed by homogeneous polynomials of degree $i+1$, for $i = 1, \ldots, M$. This completes the proof. □

The reason why each component of $\boldsymbol{W}$ is a rational function is that the solution of the corresponding homology equation can be either a polynomial or a rational vector field in the variable $\boldsymbol{x}$. In this respect, properties (ii) and (iii) of Theorem 2.6 are strong hypotheses. If (ii) is not satisfied at some order $i \geqslant 1$ then a vector field $\boldsymbol{G}_i$ is composed by rational terms. This fact could lead to the introduction of logarithmic terms in the solution of the homology equation at the next order of the process. To avoid this, one shall stop the process of the Lie transformation at the order i.

Note that Theorem 2.6 assures the construction of a formal symmetry $\boldsymbol{I}(\boldsymbol{x};\varepsilon)$ for (1.1) in the case where T and A commute. In fact,

$$\boldsymbol{I}(\boldsymbol{x};\varepsilon) = T\boldsymbol{x} + \sum_{i=1}^{M} \frac{\varepsilon^i}{i!} \mathcal{L}^i_{-\boldsymbol{W}}(T\boldsymbol{x}).$$

Thus, $\boldsymbol{I}(\boldsymbol{x};\varepsilon)$ becomes a symmetry of the initial system (with an approximation of the type $\mathcal{O}(\varepsilon^{M+1})$). Consequently, one can assure that the original system is reduced by constructing its formal symmetries. Notice that given a matrix A we always can take $T_1 = A$ and $T_2 = I_m$, where I_m is the m-dimensional identity matrix and both matrices commute with A. Thus, for non-null square matrices A, we have at least two candidates to become the linear part of the formal symmetries. Note that $\boldsymbol{I}(\boldsymbol{x};\varepsilon)$ is the approximate symmetry defined in [6].

We stress that when taking $T = A_S$ the application of Theorem 2.6 can yield a transformed system different from that obtained with Theorem 2.5. Nevertheless, in both situations $A_S\boldsymbol{y}$

becomes a linear symmetry of the transformed system. In these circumstances, the best option seems to be the application of the Normal Form Theorem. The reason is that in this way one always obtains polynomial vector fields for the transformed differential system and for the generating function. Therefore, the normal form transformation can be executed up to any order.

Remark 5. – It could be possible to construct symmetries with vanishing linear part ($T = 0$). For that, the above results should be slightly adapted but we do not enter this here. However, for the Hamiltonian case the matter is not so complicated and in [23] it is analyzed how to compute polynomial formal symmetries (integrals of motion) starting at any degree $j \geqslant 1$.

The Normal Form Theorem for the general equilibrium can be connected to Theorem 2.6 as we show in the following corollary. Indeed, we need an additional condition so that the Normal Form Theorem be a consequence of Theorem 2.6.

COROLLARY 2.7. – *Let us assume the same hypotheses as in Theorem* 2.6. *In addition to that, let us take $T = A_S$ and $A_M = A^t - A_S$. Let us apply Theorem* 2.6 *and construct the polynomial vector fields $\boldsymbol{G}_i$, $i = 1, \ldots, M$. Then, for each $i = 1, \ldots, M$, the Lie bracket $[\boldsymbol{G}_i, A^t\boldsymbol{y}]$ vanishes if and only if $\boldsymbol{G}_i \in \ker(\mathcal{L}_{A_M} \mid \mathcal{P}_i)$.*

Proof. – Taking T as above, the Lie operator associated to it is $\mathcal{L}_{A_S}$ and the vector field $A^t\boldsymbol{y}$ is decomposed as $A^t\boldsymbol{y} = T\boldsymbol{y} + A_M\boldsymbol{y}$. Now, given $\boldsymbol{G}_i \in \mathcal{Q}_i$, $\mathcal{L}_{A^t}(\boldsymbol{G}_i) = \mathcal{L}_{A_S}(\boldsymbol{G}_i) + \mathcal{L}_{A_M}(\boldsymbol{G}_i) = \mathcal{L}_{A_M}(\boldsymbol{G}_i)$, according to Theorem 2.6. Thence $[\boldsymbol{G}_i, A^t\boldsymbol{y}] = \mathbf{0}$ if and only if $[\boldsymbol{G}_i, A_M\boldsymbol{y}] = \mathbf{0}$ for each $i = 1, \ldots, M$. □

Thus, when Corollary 2.7 is satisfied Theorems 2.5 and 2.6 are equivalent.

Remark 6. – Cicogna and Gaeta [6] have given a connection between the possibility of linearizing an m-dimensional system of polynomial differential equations, $\mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t = \boldsymbol{F}(\boldsymbol{x}; \varepsilon)$, and the type of symmetries $\boldsymbol{I}_i(\boldsymbol{x}; \varepsilon)$ the system has. Specifically, these authors prove that such a system can be linearized (up to order M) if and only if it admits m independent commuting (e.g. $[\boldsymbol{I}_i, \boldsymbol{I}_j] = \mathcal{O}(\varepsilon^{M+1})$ for all $1 \leqslant i, j \leqslant m$) approximate symmetries whose linear parts have semisimple matrices T. Notice that the existence of m symmetries of this type implies that the initial system can be solved (up to an approximation of order M).

2.4. Polynomial generating functions

A particular and optimal situation appears when the solutions of the partial differential equations (2.9) are always polynomial vector fields. Then, the sets $\mathcal{R}_i$ would be subsets of $\mathcal{P}_i$ and we would never go out of polynomial domains. We have to analyze under which conditions we can obtain generating functions whose terms are vector fields formed by polynomials. To achieve that, we need first the following lemma.

LEMMA 2.8. – *Let A be a non-null $(m \times m)$-matrix with real coefficients and Jordan decomposition $A = A_S + A_N$. Let A_M be the nilpotent matrix defined as $A_M = A^t - A_S$. Let $\{\mathcal{P}_i\}_{i=0}^{M}$ be the sequence of the vector spaces of all vector fields whose components are homogeneous polynomials of degree $i + 1$ in $\boldsymbol{x} \in \boldsymbol{R}^m$. Then, the identities*:

$$\ker(\mathcal{L}_A \mid \mathcal{P}_i) = \ker(\mathcal{L}_{A_S} \mid \mathcal{P}_i) \cap \ker(\mathcal{L}_{A_N} \mid \mathcal{P}_i),$$
$$\ker(\mathcal{L}_{A^t} \mid \mathcal{P}_i) = \ker(\mathcal{L}_{A_S} \mid \mathcal{P}_i) \cap \ker(\mathcal{L}_{A_M} \mid \mathcal{P}_i)$$

and

$$\operatorname{im}(\mathcal{L}_A \mid \mathcal{P}_i) = \operatorname{im}(\mathcal{L}_{A_S} \mid \mathcal{P}_i) \oplus \big(\ker(\mathcal{L}_{A_S} \mid \mathcal{P}_i) \cap \operatorname{im}(\mathcal{L}_{A_N} \mid \mathcal{P}_i)\big),$$
$$\operatorname{im}(\mathcal{L}_{A^t} \mid \mathcal{P}_i) = \operatorname{im}(\mathcal{L}_{A_S} \mid \mathcal{P}_i) \oplus \big(\ker(\mathcal{L}_{A_S} \mid \mathcal{P}_i) \cap \operatorname{im}(\mathcal{L}_{A_M} \mid \mathcal{P}_i)\big),$$

hold for each $i = 1, \ldots, M$.

Proof. – See reference [28] (Lemma 2.2, p. 133) or [23]. □

THEOREM 2.9. – *Let us suppose the same conditions as in Theorem* 2.6. *Let* T *be an* $(m \times m)$-*matrix such that it commutes with* A. *Let us decompose* T *into its semisimple plus nilpotent parts, i.e.,* $T = T_S + T_N$ *and let us define the nilpotent matrix* $T_M = T^t - T_S$. *Then, for* $i = 1, \ldots, M$, *each term* $\boldsymbol{W}_i(\boldsymbol{x})$ *of the generating function is formed by homogeneous polynomial vector fields of degree* $i + 1$ *if and only if the intersection of the subspaces*

$$\mathrm{im}(\mathcal{L}_A \mid \mathcal{P}_i) = \big(\ker(\mathcal{L}_{A_S} \mid \mathcal{P}_i) \cap \mathrm{im}(\mathcal{L}_{A_N} \mid \mathcal{P}_i)\big) \oplus \mathrm{im}(\mathcal{L}_{A_S} \mid \mathcal{P}_i)$$

and

$$\mathrm{im}(\mathcal{L}_{T^t} \mid \mathcal{P}_i) = \big(\ker(\mathcal{L}_{T_S} \mid \mathcal{P}_i) \cap \mathrm{im}(\mathcal{L}_{T_M} \mid \mathcal{P}_i)\big) \oplus \mathrm{im}(\mathcal{L}_{T_S} \mid \mathcal{P}_i)$$

is a subspace of $\mathcal{P}_i$ *of dimension strictly positive.*

Proof. – By virtue of Lemma 2.3, we can decompose $\mathcal{P}_i$ according to the Lie operator $\mathcal{L}_{T^t} : \mathcal{P}_i \to \mathcal{P}_i$. That is, $\mathcal{P}_i = \ker(\mathcal{L}_T \mid \mathcal{P}_i) \oplus \mathrm{im}(\mathcal{L}_{T^t} \mid \mathcal{P}_i)$. Now, given $\boldsymbol{D} \in \mathcal{P}_i$ we can decompose $\boldsymbol{D}$ as the sum $\boldsymbol{D} = \boldsymbol{D}_1 + \boldsymbol{D}_2$, where $\boldsymbol{D}_1 \in \ker(\mathcal{L}_T \mid \mathcal{P}_i)$ and $\boldsymbol{D}_2 \in \mathrm{im}(\mathcal{L}_{T^t} \mid \mathcal{P}_i)$. Hypothesis (iii) of Theorem 2.6 assures that it is possible to find $\boldsymbol{E} \in \mathcal{Q}_i$ and $\boldsymbol{H} \in \mathcal{R}_i$, $i = 1, \ldots, M$, such that $\boldsymbol{E} = \boldsymbol{D} + [A\boldsymbol{y}, \boldsymbol{H}]$ and $[\boldsymbol{E}, T\boldsymbol{y}] = \boldsymbol{0}$. Thus, the search for $\boldsymbol{E}$ and $\boldsymbol{H}$ is done in two steps. First, we identify $\boldsymbol{E}$ with $\boldsymbol{D}_1$, then $\boldsymbol{H}$ has to be the solution of the partial differential identity

$$\boldsymbol{D}_2 = [\boldsymbol{H}, A\boldsymbol{y}]. \tag{2.10}$$

In principle, $\boldsymbol{H}$ is a vector field whose m components are rational functions such that the degree of the polynomial of the numerator minus the degree of the polynomial of the denominator is $i + 1$ in each component, but we can interpret equation (2.10) in terms of Linear Algebra. Indeed, $\boldsymbol{H}$ is a solution of (2.10) if and only if $\boldsymbol{D}_2 \in \mathrm{im}(\mathcal{L}_A \mid \mathcal{P}_i)$. In other words, given A and $\boldsymbol{D}_2 \in \mathrm{im}(\mathcal{L}_{T^t} \mid \mathcal{P}_i)$, the problem of finding $\boldsymbol{H}$ as a vector field whose polynomials are of degree $i + 1$ in a subset of $\mathcal{P}_i$ is equivalent to discussing whether the polynomial $\boldsymbol{D}_2$ is in the image of the set $(\mathcal{L}_A \mid \mathcal{P}_i)$. Then, $\boldsymbol{H}$ is a vector field formed by polynomials if and only if $\boldsymbol{D}_2 \in (\mathrm{im}(\mathcal{L}_A \mid \mathcal{P}_i) \cap \mathrm{im}(\mathcal{L}_{T^t} \mid \mathcal{P}_i))$.

In practice, it is convenient to obtain the decomposition of these two subspaces. By virtue of Lemma 2.8, $\mathrm{im}(\mathcal{L}_A \mid \mathcal{P}_i) = (\ker(\mathcal{L}_{A_S} \mid \mathcal{P}_i) \cap \mathrm{im}(\mathcal{L}_{A_N} \mid \mathcal{P}_i)) \oplus \mathrm{im}(\mathcal{L}_{A_S} \mid \mathcal{P}_i)$. Applying the same result to the operator T^t instead of A, we get the identity:

$$\mathrm{im}(\mathcal{L}_{T^t} \mid \mathcal{P}_i) = \big(\ker(\mathcal{L}_{T_S} \mid \mathcal{P}_i) \cap \mathrm{im}(\mathcal{L}_{T_M} \mid \mathcal{P}_i)\big) \oplus \mathrm{im}(\mathcal{L}_{T_S} \mid \mathcal{P}_i).$$

We can collect the above and conclude that the rational vector field $\boldsymbol{H}$ is a polynomial vector field whose components have degree $i + 1$ if and only if $\boldsymbol{D}_2$ belongs to the intersection of the linear subspaces $(\ker(\mathcal{L}_{A_S} \mid \mathcal{P}_i) \cap \mathrm{im}(\mathcal{L}_{A_N} \mid \mathcal{P}_i)) \oplus \mathrm{im}(\mathcal{L}_{A_S} \mid \mathcal{P}_i)$ and $(\ker(\mathcal{L}_{T_S} \mid \mathcal{P}_i) \cap \mathrm{im}(\mathcal{L}_{T_M} \mid \mathcal{P}_i)) \oplus \mathrm{im}(\mathcal{L}_{T_S} \mid \mathcal{P}_i)$. Hence, the above thesis is satisfied. □

When $T = A_S \neq 0$, the hypotheses of the Normal Form Theorem are automatically satisfied. However, if the conditions of Theorem 2.9 do not hold one has to stop the Lie transformation process at the order where logarithmic terms appear.

3. The Hamiltonian case

The theory developed in Section 2 can be translated into the symplectic context. Now, vector fields become Hamiltonian fields and formal symmetries are integrals of the transformed Hamiltonian up to a certain order. Thus, the number of degrees of freedom of a departure Hamiltonian is reduced by one after building an integral out of the transformation. Let us specify the main results.

Let $m = 2n$ and suppose that equation (1.1) can be rewritten in terms of Hamiltonian functions. Now, there is a $(2n \times 2n)$-symmetric matrix B such that $A = \mathcal{J} B$; besides, there are M vector fields $\boldsymbol{F}'_i$, such that $\boldsymbol{F}_i(\boldsymbol{x}) = \mathcal{J}\boldsymbol{F}'_i(\boldsymbol{x})$ and M scalar functions $\mathcal{H}_i(\boldsymbol{x})$, such that $\partial\mathcal{H}_i/\partial\boldsymbol{x} = \boldsymbol{F}'_i(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega \subseteq \boldsymbol{R}^{2n}$ and for all $1 \leqslant i \leqslant M$. Note that $\mathcal{J}$ denotes the skew-symmetric matrix of order $2n$, that is,

$$\mathcal{J} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Since $\boldsymbol{F}_i$ and $\boldsymbol{F}'_i$ are vector fields whose terms are homogeneous polynomials of degree $i+1$, then $\mathcal{H}_i$ represents a homogeneous polynomial in $\boldsymbol{x}$ of degree $i+2$. In this way:

$$\mathcal{H}(\boldsymbol{x}) = \mathcal{H}_0(\boldsymbol{x}) + \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i!}\mathcal{H}_i(\boldsymbol{x}), \tag{3.1}$$

where $\mathcal{H}_0(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^t B\boldsymbol{x}$.

Instead of taking a matrix T one has to choose a quadratic homogeneous polynomial, say $\mathcal{T}(\boldsymbol{x})$. Moreover, the condition of commutativity of A and T is substituted by the condition $\{\mathcal{H}_0, \mathcal{T}\} = 0$, where $\{\cdot,\cdot\}$ denotes the usual Poisson bracket of two scalar fields with symplectic structure, that is, if $\boldsymbol{x} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})^t$ and g_1 and g_2 are two scalar functions in $\boldsymbol{x}$, then:

$$\{g_1, g_2\} = \sum_{i=1}^{n}\left(\frac{\partial g_1}{\partial x_i}\frac{\partial g_2}{\partial x_{n+i}} - \frac{\partial g_1}{\partial x_{n+i}}\frac{\partial g_2}{\partial x_i}\right).$$

Theorems 2.6, 2.9 and Corollary 2.7 can be reformulated readily. Now, the polynomial $\mathcal{T}(\boldsymbol{x}) \equiv \mathcal{T}(\boldsymbol{y})$, which is an integral of $\mathcal{H}_0$, can be extended to become a (formal) integral of the transformed Hamiltonian $\mathcal{K}$ under the hypotheses of Theorem 2.6. Specifically, one has:

$$\mathcal{K}(\boldsymbol{y};\varepsilon) = \mathcal{K}_0(\boldsymbol{y}) + \sum_{i=1}^{M}\frac{\varepsilon^i}{i!}\mathcal{K}_i(\boldsymbol{y}) + \mathcal{O}\big(\varepsilon^{M+1}\big),$$

where $\mathcal{K}_0(\boldsymbol{y}) \equiv \mathcal{H}_0(\boldsymbol{x})$ and each $\mathcal{K}_i$ is a homogeneous polynomial in $\boldsymbol{y}$ of degree $i+2$ satisfying $\{\mathcal{K}_i, \mathcal{T}\} = 0$ for $i \geqslant 0$. Besides, the (truncated) change of variables $\boldsymbol{y} = \boldsymbol{Y}(\boldsymbol{x};\varepsilon)$ is given explicitly by $\boldsymbol{y} = \boldsymbol{x} + \sum_{i=1}^{M}\frac{\varepsilon^i}{i!}\mathcal{L}^i_{-\mathcal{W}}(\boldsymbol{x})$, where $\mathcal{W}$ is the scalar generating function associated to the transformation and $\mathcal{L}^i_{-\mathcal{W}}$ stands for the Lie operator constructed as the composition of $\mathcal{L}_{-\mathcal{W}}: g \to \{\mathcal{W}, g\}$ i times. Note that in principle $\mathcal{L}_{-\mathcal{W}}$ is built for scalar functions but applied to a vector field $\boldsymbol{g}$ with n components means that its image is a vector field whose components are $\mathcal{L}_{-\mathcal{W}}(g_i)$, $i = 1, \ldots, n$. Now

$$\mathcal{T}^*(\boldsymbol{x};\varepsilon) = \mathcal{T}(\boldsymbol{x}) + \sum_{i=1}^{M}\frac{\varepsilon^i}{i!}\mathcal{L}^i_{-\mathcal{W}}\big(\mathcal{T}(\boldsymbol{x})\big)$$

is a formal integral of $\mathcal{H}$ independent of it. Note that, factored by each ε^i, one has in $\mathcal{T}^*$ a homogeneous polynomial in $\boldsymbol{x}$ of degree $i+2$.

As a consequence of the introduction of $\mathcal{T}^*$ we have that the number of degrees of freedom of the initial system is reduced by one with the calculation of each integral. This is the so-called symplectic reduction. The reader can consult papers [22,23] for more details. As well, the second part of Section 5 deals with an application to a Hamiltonian system with two degrees of freedom.

4. The reduction process

4.1. The invariants associated to the reduction

From a geometrical point of view, the consequence of introducing a symmetry by making use of Theorem 2.6 is that the dimension of the phase space where the transformed system is defined (the so-called reduced phase space) is reduced from m to s (s denoting the number of functionally-independent polynomial first integrals associated to $T\boldsymbol{y}$). Let us see how this is achieved with some detail.

Fixed $\varepsilon \in \boldsymbol{R}$, the system of differential equations (1.1) is defined over an open subset of $\boldsymbol{R}^m$. This is the phase space of the dynamical system determined by (1.1). Given T an $(m \times m)$-matrix such that $AT = TA$, the application of Theorem 2.6 (after truncating at order M) leads to the vector field

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \boldsymbol{G}(\boldsymbol{y};\varepsilon) = A\,\boldsymbol{y} + \sum_{i=1}^{M} \frac{\varepsilon^i}{i!}\boldsymbol{G}_i(\boldsymbol{y}), \tag{4.1}$$

where $M \geqslant 1$ and each $\boldsymbol{G}_i(\boldsymbol{y})$ is constructed so that $[T\boldsymbol{y}, \boldsymbol{G}_i(\boldsymbol{y})] = \boldsymbol{0}$ for $1 \leqslant i \leqslant M$.

Associated to the system $\mathrm{d}\boldsymbol{y}(t)/\mathrm{d}t = T\boldsymbol{y}(t)$ there are r linearly-independent polynomial first integrals ($\varphi_i(\boldsymbol{y})$, $i = 1,\dots,r$, and $\boldsymbol{y} \in \boldsymbol{R}^m$) from which $1 \leqslant s \leqslant r$ are functionally independent. Note that $s \leqslant m$ but r can be bigger than, equal to or smaller than m. Whether $s = m$ there is no reduction in the dimension of the system. Each φ_i is a homogeneous polynomial of a certain degree $g_i \geqslant 1$. Denote by $\mathcal{L}_T^*(p(\boldsymbol{y}))$ the Lie derivative of a scalar function p associated to $T\boldsymbol{y}$, that is, $\mathcal{L}_T^*(p(\boldsymbol{y}))$ is the scalar product $(\partial p(\boldsymbol{y})/\partial \boldsymbol{y}) \cdot (T\boldsymbol{y})$. So, $\mathcal{L}_T^*(\varphi_i(\boldsymbol{y})) = 0$ for all $1 \leqslant i \leqslant r$. (Note that the φ_i are directly built from the solutions of the linear partial differential equation $\mathcal{L}_T^*(\varphi_i(\boldsymbol{y})) = 0$.)

We need to show how the (generalized) normal form transformation is effective in the sense that we really simplify the departure system. We use for that the result obtained by Gaeta in [10], adapting it to our requirements. Let $\boldsymbol{\varphi}(\boldsymbol{y}) = \{\varphi_1(\boldsymbol{y}),\dots,\varphi_r(\boldsymbol{y})\}$. Associated to these r integrals there is an $(m-s)$-dimensional Abelian Lie subgroup G_T of $GL(\boldsymbol{R}^m)$ (the Lie group of $(m \times m)$-invertible matrices with real entries), see Walcher ([32], Theorem 3.2). More precisely, G_T can be taken as $G_T = \{\exp(Tt) \in GL(\boldsymbol{R}^m) \mid t \in \boldsymbol{R}\}$. We can define a smooth mapping ϱ_T over $\boldsymbol{R}^m$:

$$\begin{aligned} \varrho_T : G_T \times \boldsymbol{R}^m &\longrightarrow \boldsymbol{R}^m \\ \big(\exp(Tt), \boldsymbol{x}\big) &\mapsto \exp(Tt)\boldsymbol{x}. \end{aligned}$$

This mapping is a natural action of G_T on $\boldsymbol{R}^m$ because it satisfies the conditions: (i) $\varrho_T(\exp(Tt_1)\exp(Tt_2), \boldsymbol{x}) = \varrho_T(\exp(Tt_1), \varrho_T(\exp(Tt_2), \boldsymbol{x}))$, $\forall t_1, t_2 \in \boldsymbol{R}$, $\forall \boldsymbol{x} \in \boldsymbol{R}^m$; and (ii) $\varrho_T(I_m, \boldsymbol{x}) = \boldsymbol{x}$ (I_m is the m-dimensional identity matrix), $\forall \boldsymbol{x} \in \boldsymbol{R}^m$. Now, there is a phase space associated to the vector field (4.1), defined as the s-dimensional quotient space $\boldsymbol{R}^m/G_T$. The reader can look up references [21,32] for details on the theoretical aspects of the reduction under the introduction of a continuous symmetry.

We take now a set of coordinates on G_T to make the splitting explicit. For that we take $\boldsymbol{\vartheta} = \{\vartheta_1, \ldots, \vartheta_{m-s}\}$ thus, the flow on G_T is indeed the time evolution of the $\vartheta_i \in G_T$. We have the following result:

THEOREM 4.1 (Splitting Lemma (Gaeta)). – *Given the generalized normal form system* (4.1) *with $\boldsymbol{G}$ a smooth function in $\boldsymbol{y}$ defined on $\boldsymbol{R}^m$, it can be transformed into a triangular system as*:

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{\varphi}(t)}{\mathrm{d}t} &= \boldsymbol{\alpha}\big(\boldsymbol{\varphi}(t);\varepsilon\big),\\ \frac{\mathrm{d}\boldsymbol{\vartheta}(t)}{\mathrm{d}t} &= \boldsymbol{\beta}\big(\boldsymbol{\vartheta}(t),\boldsymbol{\varphi}(t);\varepsilon\big),\end{aligned} \tag{4.2}$$

$\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ being smooth functions obtained constructively from $\boldsymbol{G}$.

Proof. – See reference [10]. □

Note that taking the φ_i as coordinates of the quotient space $\boldsymbol{R}^m/G_T$, the first equation of (4.2) is defined on $\boldsymbol{R}^m/G_T$ whereas the second equation of (4.2) is defined on the Lie group G_T. The vector field $\boldsymbol{\alpha}$ is constructed using the identity $\mathrm{d}\boldsymbol{\varphi}(t)/\mathrm{d}t = (\partial\boldsymbol{\varphi}/\partial\boldsymbol{y})\,\boldsymbol{G}(\boldsymbol{y};\varepsilon)$ and that the function $(\partial\boldsymbol{\varphi}/\partial\boldsymbol{y})\,\boldsymbol{G}(\boldsymbol{y};\varepsilon)$ can be expressed completely in terms of $\boldsymbol{\varphi}$ (see [10] for details). Thus we identify $\boldsymbol{\alpha}(\boldsymbol{\varphi};\varepsilon) = (\partial\boldsymbol{\varphi}/\partial\boldsymbol{y})\,\boldsymbol{G}(\boldsymbol{y};\varepsilon)$. The construction of $\boldsymbol{\beta}$ is performed once the coordinates $\boldsymbol{\vartheta}$ have been calculated. Note that as there is not a unique set of coordinates, there is not a unique function $\boldsymbol{\beta}$. Besides G_T must be a compact group, otherwise the splitting does not hold in general.

The important part of the normal form is given by the equation on $\boldsymbol{R}^m/G_T$. Moreover, if the solution of the equation involving the φ_i is known, then the solution of the remaining equation on G_T can be obtained. As there are $r-s$ functionally independent relations among the $\varphi_i(\boldsymbol{y})$, these relations are indeed the constraints determining the phase space where the (reduced) normal form in $\boldsymbol{R}^m/G_T$ is defined. Besides, the basic properties of system (4.1) are also reflected in $\boldsymbol{R}^m/G_T$. For instance, asymptotic expressions, at a certain order M, of the analytic integrals of the departure system must be found from the analysis of the normal form in $\boldsymbol{R}^m/G_T$. Other properties, as the invariance of some subsets of $\boldsymbol{R}^m$, are also preserved, formally, when passing to $\boldsymbol{R}^m/G_T$. See [32].

We have to remark that if the number of polynomial first integrals is $s = r = 0$ one still can apply the Splitting Lemma but with the drawback of loss of smoothness. Think for instance on a two-dimensional polynomial system whose linear part has the semisimple matrix $A = \mathrm{diag}\{1,1\}$. Clearly the only analytic (excepting in the axis $y_2 = 0$) first integral is $\varphi(\boldsymbol{y}) = y_1/y_2$. Once a normal form has been obtained, the explicit reduction applying Theorem 4.1 can be done similarly to what has been explained in the previous paragraphs.

Another observation refers to the possibility of performing several reductions if the number of (formal) symmetries is bigger than one, i.e., one has a vector field (4.1) for which $T_1\boldsymbol{y}, \ldots, T_p\boldsymbol{y}$ are symmetries up to an order M. In principle there is no problem of applying Theorem 4.1 p times, as the symmetries are preserved under the reduction. In practice the construction of the reduced phase spaces and appropriate functions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be cumbersome.

For Hamiltonian systems we do not need to calculate the coordinates $\boldsymbol{\vartheta}$ as the normalized Hamiltonian, by construction, is always a function depending exclusively on $\boldsymbol{\varphi}$. Besides, the reduction is done adding an extra step. First Theorem 4.1 is applied and $\boldsymbol{\varphi}$ and $\boldsymbol{\alpha}$ are calculated. Then as $\mathcal{T}$ is a new integral of $\mathcal{K}$, it is a constant of motion and one can fix a real value for it, i.e., $\mathcal{T} \equiv c \in I \subseteq \boldsymbol{R}$. Consequently, if a departure Hamiltonian defines a dynamical system on a $2n$-dimensional phase space, that is, a system of n degrees of freedom, after a symplectic reduction, the transformed Hamiltonian lies on a phase space of dimension $s-1$. Strictly speaking there is an infinite number of reduced phase spaces, one for each value of $c \in I \subseteq \boldsymbol{R}$.

Notice that different choices of T lead to different reduced dynamical systems whose flows lie on different phase spaces. In principle their relative equilibria (equilibrium points of the transformed dynamical system) would not be the same and therefore would not correspond to the same periodic orbits of the original system. Thus, performing several reductions allows us to analyze the departure dynamical system from different points of view (see for example references [22] and [23]).

4.2. The reduced phase space

We have to parameterize the reduced phase space. Indeed, the coordinates of $\boldsymbol{R}^m/G_T$ are the so-called invariants associated to the reduction process or in other words the r linearly-independent polynomial first integrals associated to T. More precisely, the invariants are defined as the generators of the ring $\mathcal{I}_T$ of ϱ_T-invariant functions:

$$\mathcal{I}_T = \left\{\varphi_i(\boldsymbol{y}) \in \mathcal{P}_{g_i-1} \mid \mathcal{L}_T^*\big(\varphi_i(\boldsymbol{y})\big) = 0,\ i = 1, \ldots, r\right\},$$

such that each $\varphi_i \in \mathcal{P}_{g_i-1}$ is a homogeneous polynomial in $\boldsymbol{y}$ of a certain degree g_i with $g_i > 0$. In the Hamiltonian case one can still encounter non-trivial invariant polynomials of degree one. As pointed out before, the dimension of $\boldsymbol{R}^m/G_T$ is s thus, there are s functionally independent invariants. However, the number r of linearly independent invariants cannot be obtained in a systematic manner, and it depends on each reduction, that is, it is determined by the choice of the matrix T, but it satisfies the inequality $r \geqslant s$. Note that there must be at least $r-s$ polynomial relations involving the φ_i. These relations define the reduced phase space.

This space can have singular points due to the existence of non-trivial isotropy subgroups. Specifically, given the Lie group G_T associated to the matrix T and its natural action ϱ_T on $\boldsymbol{R}^m$, the isotropy subgroup of a vector $\boldsymbol{x} \in \boldsymbol{R}^m$ is defined as:

$$G_T^{\boldsymbol{x}} = \left\{\exp(Tt) \in G_T \mid \varrho_T\big(\exp(Tt), \boldsymbol{x}\big) = \boldsymbol{x}\right\}.$$

Now, if for all $\boldsymbol{x} \in \boldsymbol{R}^m$ the isotropy subgroup of $\boldsymbol{x}$ is trivial, the reduced phase space is a smooth manifold. This is the so-called regular reduction [17]. On the contrary, if there is an $\boldsymbol{x} \in \boldsymbol{R}^m$ such that its isotropy subgroup is non-trivial, the reduced phase space is a manifold with singularities. That reduction is called singular [1].

If the reduction is symplectic there is another possibility of introducing singularities. Indeed, after determining the corresponding invariants and computing the reduced Hamiltonian up to the desired order, the value of $\mathcal{T}$ has to be fixed to a constant $c \in \boldsymbol{R}$. This constant appears as a parameter in the constraints which define the reduced phase spaces. In other words, one has a parametric family of reduced phase spaces with at least one parameter (the constant c). Thus, these reduced phase spaces can have different number of singularities depending on the values the parameter c takes. These situations cannot be detected by analyzing the corresponding isotropy subgroups. The best way of calculating the singularities consists in parameterizing first the equation which defines the reduced phase spaces and computing thereafter their gradient vectors. The singularities are those points where the gradient vanishes.

4.3. Implementation

We describe how to perform a transformation based on the theory exposed in Sections 2, 3 and Subsection 4.1. Suppose we are given a positive integer M, two matrices A and T and M vector fields $\boldsymbol{F}_j$, $1 \leqslant j \leqslant M$. Moreover, at a certain order i of the transformation process, the vector fields $\boldsymbol{W}_j$ and $\boldsymbol{G}_j$ are known for $1 \leqslant j \leqslant i-1$ and $\widetilde{\boldsymbol{F}}_i$ is also known. Let us specify how to solve the homology equation (2.9) or, in other words, how to obtain $\boldsymbol{W}_i$ and $\boldsymbol{G}_i$.

First of all, a previous linear transformation of the initial system in order to bring the matrix A into its Jordan form J is advisable. By doing it, the linear part of the system is reduced to its simplest expression, therefore its corresponding Lie operator $\mathcal{L}_J$ contains less terms than $\mathcal{L}_A$ which implies that the associated homology equation could be solved in an easier way. For Hamiltonian systems, this previous transformation must be done with care in order not to destroy the symplectic structure of the equation, see [16] for a thorough study of this.

As a second step, according to Subsections 2.3 and 2.4 we have to choose $\boldsymbol{G}_i = \widetilde{\boldsymbol{F}}_i^{\#}$, where $\widetilde{\boldsymbol{F}}_i^{\#} \in \ker(\mathcal{L}_T)$. Thus, in order to calculate $\boldsymbol{G}_i$ we have to express it as a vector field with homogeneous polynomial components of degree $i+1$ in $\boldsymbol{y}$. Now, the coefficients of these polynomials must be determined. In order to calculate them we make a matching with the coefficients of the terms appearing in $\widetilde{\boldsymbol{F}}_i$. Thus, $\boldsymbol{G}_i$ is determined by solving a system of linear equations (the unknowns being the coefficients of the polynomials). The expression of $\boldsymbol{G}_i$ is not unique, in general; one extra condition is needed, see reference [20] (Theorem 4, pp. 181–182).

The next step is the calculation of $\boldsymbol{W}_i$ by solving the system of (linear) partial differential equations (2.9). Recall that at this point $\widetilde{\boldsymbol{F}}_i^{\&}$ has been obtained. In the cases where matrix A has non-null semisimple part we have two alternatives to obtain a formal symmetry: either applying the Normal Form Theorem or Theorem 2.6. Let us note that in the cases where A is nilpotent only the second way is possible. When applying the Normal Form Theorem, we know that the term of the generating function $\boldsymbol{W}_i$ is a vector field composed by m polynomials of degree $i+1$. Therefore, equation (2.9) can be resolved by matching the coefficients of m arbitrary homogeneous polynomials of degree $i+1$ with the coefficients of the terms of $\widetilde{\boldsymbol{F}}_i^{\&}$. In the case of Theorem 2.6, the reduction can be more effective if one chooses an adequate set of variables and solves equation (2.9) directly. In general, the resolution of (2.9) is not an easy task but the built-in function DSOLVE of MATHEMATICA [35] is rather useful in most of the cases. Other alternatives are based on the choice of appropriate sets of variables in which the partial differential quantity $\mathcal{L}_A(\boldsymbol{W}_i)$ is expressed in an "easier" way. See some examples of Hamiltonian reductions in cf. [24]. We refer also to paper [33] for a systematic way of calculating the vector fields $\boldsymbol{G}_i$ and $\boldsymbol{W}_i$ in each step, when the Normal Form Theorem is applied.

Concerning the calculation of each $\widetilde{\boldsymbol{F}}_i$, it is based on the algorithm of Lie transformations. The reader is referred to cf. [19] for more details.

Remark 7. – If the departure dynamical system can be split into terms of Hamiltonian nature and others of non-Hamiltonian nature, one can take advantage of this as the type of operations involved for Hamiltonian systems is less cumbersome than its equivalent in vector fields. For instance, the system of partial differential equations to be solved for the general case (for the homology equation) reduces to a unique (at each order of the Lie process) partial differential equation. Besides, the Lie brackets are converted into Poisson brackets, which helps to save a considerable computational cost (up to 25%). To achieve this, one has to adapt the corresponding algorithms splitting the Lie operator $\mathcal{L}_A$ into two parts: one of vectorial nature and the other of scalar nature. An algorithm concerning this approach can be found in cf. [3].

In the next step the splitting of the normal form system $\boldsymbol{G}$ must be performed. At this point we have already computed the r first integrals associated to the linear system $\mathrm{d}\boldsymbol{y}(t)/\mathrm{d}t = T\boldsymbol{y}(t)$. After obtaining the Lie group G_T and a coordinate set $\boldsymbol{\vartheta}$ on it, we calculate the vector fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Note that $\boldsymbol{\alpha}$ is uniquely determined but $\boldsymbol{\beta}$ is not. As said before, for Hamiltonians one always can take $\boldsymbol{\beta} = \mathbf{0}$ as the symplectic change $\boldsymbol{x} \to \boldsymbol{y}$ converts a Hamiltonian $\mathcal{H}(\boldsymbol{x})$ into a Hamiltonian $\mathcal{K}(\boldsymbol{y})$ which can be rewritten as $\mathcal{K}(\varphi_1, \ldots, \varphi_r)$ (perhaps with $r > m$, see cf. [36]

for examples of this). Finally it remains to determine the reduced phase space from the $r - s$ relations among the φ_i.

5. Applications

We apply the theory of Sections 1, 2, 3 and 4 to some examples.

5.1. Three-dimensional nilpotent matrices

One of the applications of the theory developed before concerns the cases of polynomial dynamical systems whose linear parts have nilpotent real matrices. In these situations the application of the Normal Form Theorem does not produce a new formal symmetry. The two-dimensional case has been treated in [25]. Here we deal with (3×3)-matrices with real entries.

Consider the system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = A\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \tag{5.1}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)^t \in \boldsymbol{R}^3$ and A is either

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us suppose that the vector field $\boldsymbol{f}(\boldsymbol{x})$ has three components $f_1(\boldsymbol{x})$, $f_2(\boldsymbol{x})$ and $f_3(\boldsymbol{x})$ corresponding to arbitrary Taylor series in $\boldsymbol{x}$ starting at degree two. Clearly A_1, A_2 and A_3 are nilpotent since $A_1^2 = A_3^2 = 0$ and $A_2^3 = 0$. Systems (5.1) are studied from the Stability Theory point of view with the aim of analyzing if the origin can be stable. Besides, scalar equations of the form $\mathrm{d}^3x/\mathrm{d}t^3 + f(x, \mathrm{d}x/\mathrm{d}t, \mathrm{d}^2x/\mathrm{d}t^2)$, where f has a Taylor expansion starting at degree two, can be written in the form (5.1) with $A = A_2$. Because of symmetric considerations we study (5.1) with $A = A_1$ and A_2, as the case A_3 can be readily inferred from the analysis for A_1.

Note that due to the form taken by the function $\boldsymbol{f}$ we have the freedom of calculating the normal forms and the generating functions in a compact manner, which allows to simplify the notations and calculations. Besides, the Lie transformations are executed easily to any order and in one step. In a real application we should cut the Taylor expansions at an order M but the rest of the formulae apply straightforwardly. In addition to this, we should scale the system defined by (5.1), say $\boldsymbol{x} \to \varepsilon\boldsymbol{x}'$, so as to introduce a dimensionless small parameter $\varepsilon > 0$. In this manner the equation would appear in the appropriate setting to apply a perturbation theory. However we can avoid this step as we do the Lie transformation in one step. Thus from now on we can fix the value of ε, that is, without loss of generality we make $\varepsilon = 1$.

First of all we apply the Normal Form Theorem. Since $A = A_N$ and $A_S = 0$ (for both A_1 and A_2), no symmetry is going to appear as a consequence of this transformation and therefore the Splitting Lemma does not apply. Note that they are the only matrices (and their Jordan-equivalent) in three dimensions whose semisimple part is zero. More concretely, equations (5.1) are converted into:

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = A\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}), \tag{5.2}$$

with $\boldsymbol{y} = (y_1, y_2, y_3)^t$, $\boldsymbol{g} = (g_1, g_2, g_3)^t$ and

$$g_1(\boldsymbol{y}) = \alpha(y_1, y_2), \quad g_2(\boldsymbol{y}) = y_2\beta(y_1, y_2), \quad g_3(\boldsymbol{y}) = y_3\beta(y_1, y_2) + \gamma(y_1, y_2),$$

for $A = A_1$ whereas for $A = A_2$,

$$g_1(\boldsymbol{y}) = \alpha(y_1), \quad g_2(\boldsymbol{y}) = \frac{y_2}{y_1}\alpha(y_1) + \beta(y_1),$$
$$g_3(\boldsymbol{y}) = \frac{y_2^2}{2y_1^2}\alpha(y_1) + \frac{y_2}{y_1}\beta(y_1) + \gamma\big(y_1, 2y_1y_3 - y_2^2\big).$$

For the choice $A = A_1$ the Taylor series of $\alpha(y_1, y_2)$ and $\gamma(y_1, y_2)$ start at degree two and the Taylor series of $\beta(y_1, y_2)$ starts at degree one. For $A = A_2$ the Taylor series $\alpha(y_1)$, $\beta(y_1)$, $\gamma(y_1, 2y_1y_3 - y_2^2)$ start at degree two. So, in all the cases the vector field $\boldsymbol{g}$ has polynomial components in $\boldsymbol{y}$ starting at degree two. The corresponding generating functions are also polynomial as we have made use of the Normal Form Theorem. Because systems (5.2) have been constructed through the application of the Normal Form Theorem, then $[A^t\boldsymbol{y}, \boldsymbol{g}(\boldsymbol{y})] = \boldsymbol{0}$. As the two systems (5.1) and (5.2) are defined over $\boldsymbol{R}^3$, their reduced phase spaces coincide although the transformed systems are simpler than the original ones.

As a second choice we take $T = A_1$ and $T = A_2$, respectively. Note that there are other matrices commuting with A_1 and A_2 but here we only focus on the determination of formal symmetries with $T = A$. Now, we have to solve $\mathcal{L}_A(\boldsymbol{w}) + \boldsymbol{g} = \boldsymbol{f}$, where $\boldsymbol{g} \in \ker(\mathcal{L}_T)$ and $\boldsymbol{w}$ is a solution of $\mathcal{L}_A(\boldsymbol{w}) = \boldsymbol{f} - \boldsymbol{g}$. The application of Theorem 2.6 yields the reduced system:

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = A\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}), \tag{5.3}$$

where for $A = A_1$,

$$g_1(\boldsymbol{y}) = \alpha(y_1, y_3), \quad g_2(\boldsymbol{y}) = y_2\beta(y_1, y_3) + \gamma(y_1, y_3), \quad g_3(\boldsymbol{y}) = y_3\beta(y_1, y_3), \tag{5.4}$$

and for $A = A_2$,

$$\begin{aligned} g_1(\boldsymbol{y}) &= \frac{y_1}{y_3}\alpha(y_3) + \frac{y_2}{y_3}\beta(y_3) + \gamma\big(2y_1y_3 - y_2^2, y_3\big), \\ g_2(\boldsymbol{y}) &= \frac{y_2}{y_3}\alpha(y_3) + \beta(y_3), \quad g_3(\boldsymbol{y}) = \alpha(y_3). \end{aligned} \tag{5.5}$$

When $A = A_1$ the Taylor series of $\alpha(y_1, y_2)$ and $\gamma(y_1, y_3)$ start at degree two and the Taylor series of $\beta(y_1, y_3)$ at degree one. When $A = A_2$ the Taylor series $\alpha(y_3)$, $\beta(y_3)$, $\gamma(2y_1y_3 - y_2^2, y_3)$ start at degree two. Again, in all the cases the vector field $\boldsymbol{g}$ has homogeneous polynomial components in $\boldsymbol{y}$ starting at degree two.

Now, the generating function is rational because Theorem 2.9 does not apply. Specifically, for $A = A_1$,

$$w_1(\boldsymbol{y}) = -\frac{y_2}{y_3}\alpha(y_1, y_3) + \frac{1}{y_3}\int f_1(\boldsymbol{y})\,\mathrm{d}y_2,$$
$$w_2(\boldsymbol{y}) = -\frac{y_2^2}{y_3}\beta(y_1, y_3) - \frac{y_2}{y_3}\gamma(y_1, y_3) + \frac{1}{y_3}\int f_2(\boldsymbol{y})\,\mathrm{d}y_2 + \frac{1}{y_3^2}\int\left(\int f_3(\boldsymbol{y})\,\mathrm{d}y_2\right)\mathrm{d}y_2,$$
$$w_3(\boldsymbol{y}) = -y_2\,\beta(y_1, y_3) + \frac{1}{y_3}\int f_3(\boldsymbol{y})\,\mathrm{d}y_2.$$

For $A = A_2$, the expression for $\boldsymbol{w}(\boldsymbol{y})$ is more involved. Indeed, it is not possible to give an explicit formula in terms of a general vector field $\boldsymbol{f}$. Hence, one needs to substitute $\boldsymbol{f}$ in terms of polynomials starting at degree two. We have done it with MATHEMATICA but for an arbitrary polynomial vector field $\boldsymbol{f}$ of degree two and with three components; the resulting expression for $\boldsymbol{w}$ is quite big. For the two choices of A, $\boldsymbol{w}$ is a rational function having y_3 in the denominators. Thus the reductions are not defined if $y_3 = 0$. From this point of view, the open domain (subset of $\boldsymbol{R}^3$) which has to be chosen to define the transformation must exclude the line $y_3 = 0$. This makes the normal forms useless for analyzing the origin. However, it is also possible to use (5.3) in other points of the corresponding reduced phase space.

Note that $[T\boldsymbol{y}, A\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y})] = \boldsymbol{0}$ for both normal forms. Therefore $T\boldsymbol{y}$ is a symmetry of the transformed systems, up to a certain order, and we can apply Theorem 4.1. We obtain two functionally-independent first integrals in both cases. For $A = A_1$ one has $\varphi_1(\boldsymbol{y}) = y_1$ and $\varphi_2(\boldsymbol{y}) = y_3$ whereas for $A = A_2$, $\varphi_1(\boldsymbol{y}) = 2y_1y_3 - y_2^2$ and $\varphi_2(\boldsymbol{y}) = y_3$. In both cases we have $r = s = 2$ and then $m - s = 1$.

For $A = A_1$ an adequate choice of ϑ (the coordinate associated to the Lie group G_T) consists in identifying it with y_2. The reason is that φ_1 and φ_2 are precisely y_1 and y_3. Thus, equation (5.4) becomes the polynomial system:

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = \alpha(\varphi_1, \varphi_2), \qquad \frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = \varphi_2\,\beta(\varphi_1, \varphi_2). \tag{5.6}$$

The remaining one-dimensional system is defined by the polynomial system:

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \varphi_2 + \gamma(\varphi_1, \varphi_2) + \beta(\varphi_1, \varphi_2)\,\vartheta.$$

Note that the second equation is linear in ϑ. Besides, the dynamics (existence of equilibria, periodic trajectories and asymptotic expressions of the analytic first integrals) of the initial system (5.1) can be analyzed in equation (5.6), excepting in the axis $y_3 = 0$.

For $A = A_2$ we can make $\vartheta = y_2$ (we also could have chosen $\vartheta = y_1$). Thus, the splitting is as follows:

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = \frac{2\varphi_1}{\varphi_2}\,\alpha(\varphi_2) + 2\varphi_2\,\gamma(\varphi_1, \varphi_2), \qquad \frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = \alpha(\varphi_2), \tag{5.7}$$

whereas the one-dimensional equation reads as:

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \varphi_2 + \beta(\varphi_2) + \frac{\alpha(\varphi_2)}{\varphi_2}\,\vartheta.$$

Note that the second equation is linear in ϑ and both systems are polynomial in φ_1, φ_2. On this occasion, except for the axis $y_3 = 0$, we can analyze system (5.7) to infer qualitative properties of the departure system (5.1).

The Lie group associated to each T is the one-dimensional set $G_T = \{\exp(Tt) \in GL(\boldsymbol{R}^3) \mid t \in \boldsymbol{R}\}$, where for $T = A_1$ and $T = A_2$ we have respectively:

$$\exp(Tt) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(Tt) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

We define the natural action:

$$\begin{aligned} \varrho_T : G_T \times \big(\boldsymbol{R}^3 \setminus \{y_3 = 0\}\big) &\longrightarrow \boldsymbol{R}^3 \setminus \{y_3 = 0\} \\ \big(\exp(Tt), \boldsymbol{y}\big) &\mapsto \exp(Tt)\,\boldsymbol{y}. \end{aligned}$$

These mappings are natural actions of G_T on $\boldsymbol{R}^3 \setminus \{y_3 = 0\}$. Thus, systems (5.6) and (5.7) are defined over $(\boldsymbol{R}^3 \setminus \{y_3 = 0\})/G_T$, which are the reduced phase spaces. As for the two choices of T, the corresponding φ_1 runs over $\boldsymbol{R}$ whereas φ_2 runs over $\boldsymbol{R} \setminus \{0\}$, then both reduced phase spaces can be identified with $\boldsymbol{R} \times (\boldsymbol{R} \setminus \{0\})$, that is, $(\boldsymbol{R}^3 \setminus \{y_3 = 0\})/G_T \cong \boldsymbol{R} \times (\boldsymbol{R} \setminus \{0\})$. This time, the transformation has permitted to reduce the dimension of the phase space by one.

Notice that if one is interested in studying the initial system (5.1) in a vicinity of $y_3 = 0$ by means of normal form calculations, the only way is to resort to the analysis of system (5.2) in $\boldsymbol{R}^3$.

Finally, the isotropy subgroups are trivial for all $\boldsymbol{y} \in \boldsymbol{R}^3 \setminus \{y_3 = 0\}$. This implies that both reductions are regular in this subset of $\boldsymbol{R}^3$.

5.2. Hamiltonian case: a non-null semisimple matrix

Let us consider a Hamiltonian system $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, where

$$\mathcal{H}_0(\boldsymbol{x}) = \pm\tfrac{1}{2} X^2 + \tfrac{1}{2}\big(Y^2 + \omega^2 y^2\big), \tag{5.8}$$

and $\mathcal{H}_1$ a polynomial in $\boldsymbol{x} = (x, y, X, Y)$ of degree three or bigger. Variables x and y stand for the coordinates whereas X and Y represent their associated moments. Besides, $\omega \neq 0$ is a frequency. Hamiltonian $\mathcal{H}$ defines a dynamical system with two degrees of freedom. More precisely, it has been used to describe versal unfoldings of this equilibrium point. The codimension of the singularity produced by the double-zero eigenvalue is one. This means that the unfoldings can be met generically if at least one parameter is present. It also has been studied in references [23,24].

The matrix A associated to $\mathcal{H}_0$ has eigenvalues $\pm\sqrt{-1}\,\omega^2$ and zero (double). Concretely, it is given as:

$$A = \begin{pmatrix} 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \end{pmatrix},$$

with semisimple and nilpotent parts given, respectively, by:

$$A_S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \end{pmatrix}, \qquad A_N = \begin{pmatrix} 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By means of a reduction process, Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ can be transformed into an equivalent one but with one degree of freedom. There are several alternatives to achieve the reduction. Before, we need to be more specific about how to solve the homology equation (2.9).

First of all we perform the (symplectic) change of variables:

$$v = \frac{1}{\sqrt{2}}\left(y - \frac{\sqrt{-1}}{\omega} Y\right), \qquad V = \frac{1}{\sqrt{2}}\big(Y - \sqrt{-1}\,\omega y\big).$$

Then, Hamiltonian $\mathcal{H}_0$ is transformed into

$$\mathcal{H}_0(x, v, X, V) = \pm\tfrac{1}{2} X^2 + \sqrt{-1}\,\omega v V$$

whereas the perturbation is formed by monomials of the type $cx^j v^k X^\ell V^m$, with c a real or complex constant. In these variables, the homology equation (2.9) reads as:

$$\mp X\frac{\partial\mathcal{W}_i}{\partial x}+\sqrt{-1}\,\omega\left(v\frac{\partial\mathcal{W}_i}{\partial v}-V\frac{\partial\mathcal{W}_i}{\partial V}\right)+\mathcal{K}_i=\widetilde{\mathcal{H}}_i, \tag{5.9}$$

where $\widetilde{\mathcal{H}}_i$ refers to the “Hamiltonian version” of $\widetilde{\boldsymbol{F}}_i$ in (2.9). The function DSOLVE of the manipulator MATHEMATICA successes in solving this equation for a given monomial. Specifically the antiimage of the monomial $x^j v^k X^\ell V^m$ is given by:

$$\begin{aligned}&\mathcal{W}_i^{(j,k,\ell,m)}(x,v,X,V)\\&\quad=\left[\frac{\sqrt{-1}}{\omega(k-m)}\right]^{j+1}x^j v^k X^\ell V^m\sum_{n=0}^{j}\frac{j!}{n!}\big[-\sqrt{-1}\,\omega(k-m)\big]^n\left(\frac{x}{\pm X}\right)^{n-j},\end{aligned} \tag{5.10}$$

if $k\neq m$ and

$$\mathcal{W}_i^{(j,k,\ell,m)}(x,v,X,V)=\frac{\mp 1}{j+1}x^{j+1}X^{\ell-1}(vV)^k, \tag{5.11}$$

if $k=m$. One has to notice that the function $\mathcal{W}_i^{(j,k,\ell,m)}$ is rational with powers of X in some of its denominators. Indeed, when $k=m$ negative powers of X appear if and only if $\ell\leqslant 0$, whereas if $k\neq m$ then $\mathcal{W}_i^{(j,k,\ell,m)}$ contains negative powers of X if and only if $\ell<0$. The powers of x, v and V are always zero or positive integers. Observe that although the original Hamiltonian is polynomial, negative powers of X are first introduced in $\mathcal{W}$ because of the terms where $\ell=0$.

Now, we are in conditions of using the results of Sections 2, 3 and 4. We use two different choices for T giving rise to two different reductions of the same Hamiltonian.

First we apply the Normal Form Theorem. Note that, since $A_N\neq 0$, the transformed Hamiltonian is going to be of one degree of freedom. Now, it can be proven that Corollary 2.7 applies. Therefore, if one takes $T=A_S$, the solution given by the application of Theorem 2.1 coincides with the solution of the Normal Form Theorem.

After truncating at order M, the normal form of $\mathcal{H}$ is given by:

$$\mathcal{K}(x,v,X,V)=\pm\tfrac{1}{2}X^2+\sqrt{-1}\,\omega vV+\varepsilon\mathcal{K}_1(x,X,vV)+\cdots+\frac{\varepsilon^M}{M!}\mathcal{K}_M(x,X,vV).$$

With this choice of T, the polynomial related to the semisimple part of $\mathcal{H}_0$, that is, $\mathcal{T}=\frac{1}{2}(Y^2+\omega^2y^2)\equiv\sqrt{-1}\,\omega vV$ becomes an integral of the transformed Hamiltonian, that is, $\{\mathcal{K}_i,\mathcal{T}\}=0$, for $0\leqslant i\leqslant M$. Now, the Lie operator associated to T (or to $\mathcal{T}$ as we are in a Hamiltonian case) is:

$$\mathcal{L}_T(\mathcal{W}_i)=\sqrt{-1}\,\omega\left(v\frac{\partial\mathcal{W}_i}{\partial v}-V\frac{\partial\mathcal{W}_i}{\partial V}\right).$$

Thus, the reason why the variables v and V must appear in $\mathcal{K}_i$ as powers of (vV) is that $x^j v^k X^\ell V^m$ belongs to $\ker(\mathcal{L}_T)$ if and only if $k=m$. Note that saying that $\mathcal{T}$ is an integral of $\mathcal{K}$ is equivalent to say that $T\boldsymbol{y}$ is a symmetry of the truncated problem. Finally, $\mathcal{W}_i$ is a polynomial function since no negative powers of X are introduced in the process. One can see that hypotheses of Theorem 2.3 hold.

Two invariants (polynomial first integrals) of the reduction can be taken linear. The other is a quadratic polynomial. Explicitly they are:

$$\varphi_1 = x, \quad \varphi_2 = X, \quad \varphi_3 = \tfrac{1}{2}\big(Y^2 + \omega^2 y^2\big).$$

As they are functionally and linearly independent then $r = s = 3$. Besides, the transformed Hamiltonian can be expressed in terms of them, i.e., $\mathcal{K} \equiv \mathcal{K}(\varphi_1, \varphi_2, \varphi_3)$. Note that as we are able to express the normal form system in terms of the invariants then, in terms of the Splitting Lemma, the vector field $\boldsymbol{\beta} = \mathbf{0}$ and the dynamics is totally expressed in terms of $\boldsymbol{\alpha}$, or equivalently, in terms of $\varphi_1, \varphi_2, \varphi_3$.

Fixing $\mathcal{T} \equiv c \in [0, +\infty)$ the reduced phase spaces have dimension $s - 1 = 2$ and are the planes $\varphi_3 = c$ in the three-dimensional frame formed by φ_1, φ_2 and φ_3.

The exponential matrix $\exp(T\,t)$ is given by:

$$\exp(Tt) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\omega t) & 0 & \frac{1}{\omega}\sin(\omega t) \\ 0 & 0 & 1 & 0 \\ 0 & -\omega\sin(\omega t) & 0 & \cos(\omega t) \end{pmatrix}.$$

Thus, the Lie group G_T associated to T has dimension one as the dimension of the system is four and $s = 3$. We define now the natural action:

$$\begin{aligned} \varrho_T : G_T \times \boldsymbol{R}^4 &\longrightarrow \boldsymbol{R}^4 \\ \big(\exp(Tt), \boldsymbol{x}\big) &\mapsto \exp(Tt)\boldsymbol{x}. \end{aligned}$$

It can be deduced from the above that there is no nontrivial isotropy subgroup. Therefore the reduction is regular.

As a second possibility we take $T = A_N$. Now, we have to perform a transformation such that $\boldsymbol{T}\,\boldsymbol{y}$ will become a symmetry of the truncated problem. In other words, $\mathcal{K}$ will be constructed so that the part of $\mathcal{H}_0$ associated to A_N ($\mathcal{T} = \pm\frac{1}{2}X^2$) will become an integral of $\mathcal{K}$. But this is equivalent to say that x must not appear in $\mathcal{K}$. Note that the Lie operator of T is

$$\mathcal{L}_T(\mathcal{W}_i) = \mp X \frac{\partial \mathcal{W}_i}{\partial x}.$$

Hence, the transformed Hamiltonian reads now, after truncation:

$$\mathcal{K}(x, v, X, V) = \pm\tfrac{1}{2}X^2 + \sqrt{-1}\,\omega v V + \varepsilon\mathcal{K}_1(X, v, V) + \cdots + \frac{\varepsilon^M}{M!}\mathcal{K}_M(X, v, V).$$

This time the generating function is no longer polynomial as Theorem 2.3 does not apply. The reason is that when dealing with the homology equation (5.9), in $\widetilde{\mathcal{H}}_i$ there are some monomials $x^j\,v^k\,X^\ell\,V^m$ with $\ell = 0$ and $k = m$. Hence, according to (5.11), their antiimages $\mathcal{W}_i^{(j,k,\ell,m)}(x, v, X, V)$ have the exponent -1 in the powers of X. From this, and by virtue of (5.10) and (5.11), the exponents $-1, -2, \ldots,$ appear in the next orders of $\mathcal{W}$. Therefore, $\mathcal{W}$ is rational and the line $X = 0$ is removed from the domain of definition of the transformation.

The invariants associated to the reduced system can be taken linear. They are:

$$\varphi_1 = X, \quad \varphi_2 = y, \quad \varphi_3 = Y$$

and the transformed Hamiltonian can be expressed in terms of them, i.e., $\mathcal{K} \equiv \mathcal{K}(\varphi_1, \varphi_2, \varphi_3)$. Again $r = s = 3$ and according to the application of Theorem 4.1 to symplectic systems, the vector field $\boldsymbol{\beta} = \mathbf{0}$ and the normal form is completely defined as a function of $\boldsymbol{\alpha}$, i.e., as a function of $\varphi_1, \varphi_2, \varphi_3$.

Fixing $c \in \boldsymbol{R} \setminus \{0\}$, the reduced phase spaces are of dimension $s - 1 = 2$. Specifically, they are the planes $\varphi_1 = c$ in the three-dimensional frame defined by φ_1, φ_2 and φ_3.

We have to see now if there are nontrivial isotropy subgroups related to the reduction. The exponential matrix $\exp(T t)$ is:

$$\exp(Tt) = \begin{pmatrix} 1 & 0 & \pm t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thence, the Lie group is again one-dimensional with natural action:

$$\begin{aligned} \varrho_T : G_T \times \big(\boldsymbol{R}^4 \setminus \{X = 0\}\big) &\longrightarrow \boldsymbol{R}^4 \setminus \{X = 0\} \\ \big(\exp(Tt), \boldsymbol{x}\big) &\mapsto \exp(Tt)\boldsymbol{x}. \end{aligned}$$

There is no $\exp(Tt) \in G_T \setminus \{I_4\}$ with $\varrho_T(\exp(Tt), \boldsymbol{x}) = \boldsymbol{x}$ for $\boldsymbol{x} \in \boldsymbol{R}^4 \setminus \{X = 0\}$. The reduction is therefore regular.

Note that we could apply the two transformations explained in this subsection consecutively, that is, passing from the initial Hamiltonian defined by $\mathcal{H}$ to a normalized Hamiltonian $\mathcal{S}$ for which X and $\frac{1}{2}(Y^2 + \omega^2 y^2)$ are formal integrals of it. Then, $\mathcal{S}$ would define a system of zero degrees of freedom with trivial flow.

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